Perturbative analysis of non-singular cosmological model[1]

Shinsuke KAWAI[2]

Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606, Japan

Masa-aki SAKAGAMI[3] and Jiro SODA[4]

Department of Fundamental Sciences, FIHS, Kyoto University, Kyoto 606, Japan

Abstract

A stability analysis is made for a non-singular pre-big-bang like cosmological model based on 1-loop corrected string effective action. Its homogeneous and isotropic solution realizes non-singular transition from de Sitter universe to Friedmann-like universe, via super inflation phase. We are interested in whether the non-singular nature of the solution would be stable or not in more general inhomogeneous case. Perturbative analysis is made for scalar, vector, and tensor linear perturbations, and instability is found for tensor-type perturbation.

1 Introduction

One of the most remarkable predictions of General Relativity is the existence of singularities. Particularly at the center of the black hole and in the very early universe, one might interpret the singularities as a result of the breakdown of General Relativity. If so, the behavior of the system "near" the singularity should be explained by some more general theory, preferably including GR as an effective theory in particular situation. In this sense, searches for singularity-free cosmological models have been made by many authors [8]. Recently non-singular cosmological models based on superstring theory[1] were presented, and among these, we will here investigate the model presented in [13], which is derived from the superstring effective action with 1-loop string correction [11]

$$\mathcal{S} = \int d^4x\sqrt{-g}\{\frac{1}{2}R - \frac{1}{4}(D\Phi)^2 - \frac{3}{4}(D\sigma)^2 + \frac{1}{16}[\lambda_1 e^{\Phi} - \lambda_2\xi(\sigma)]R^2_{GB}\} \tag{1}$$

where R ,Φ and σ are the Ricci scalar curvature, the dilaton, and the modulus field, respectively. The Gauss-Bonnet curvature is

$$R^2_{GB} = R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda} - 4R^{\mu\nu}R_{\mu\nu} + R^2, \tag{2}$$

and $\xi(\sigma)$ is a function of modulus field expressed with Dedekind η function:

$$\xi(\sigma) = -\ln[2e^{\sigma}\eta^4(ie^{\sigma})]. \tag{3}$$

Since the non-singular nature essentially depends on the behavior of modulus field σ, we will concentrate on the modulus field only and actually use the effective action appearing in [13]:

$$\mathcal{S} = \int d^4x\sqrt{-g}\{\frac{1}{2}R - \frac{1}{2}(D\varphi)^2 - \frac{\lambda}{16}\xi(\varphi)R^2_{GB}\}. \tag{4}$$

From this action we will derive the back ground equations of motion and see the behavior of the solution in the next section. In the section 3, the method of perturbation is explained and equations of motion for perturbative variables are derived, and also the stability of this cosmological model is discussed. The results are summarized in the section 4.

[1] Talk given at YITP, Kyoto, in the VIIth Workshop on General Relativity and Gravitation (Oct.27-30, 1997).
[2]E-mail:kawai@phys.h.kyoto-u.ac.jp
[3]E-mail:sakagami@phys.h.kyoto-u.ac.jp
[4]E-mail:jiro@phys.h.kyoto-u.ac.jp

2 Model and the background solution

We will assume the homogeneous and isotropic metric with conformal time η,

$$ds^2 = a(\eta)^2(-d\eta^2 + \gamma_{ij}dx^i dx^j), \tag{5}$$

$$\gamma_{ij} = \frac{1}{1+\frac{1}{4}\mathcal{K}(x^2+y^2+z^2)}\delta_{ij} \tag{6}$$

where $\mathcal{K} = 0, +1, -1$ for flat, closed, open universe, respectively. From the action (4) one can derive the equations of motion as

$$\varphi'^2 = 6(\mathcal{H}^2+\mathcal{K})(1-\frac{\lambda}{2a^2}\mathcal{H}\xi') \tag{7}$$

$$(\mathcal{H}'+\mathcal{H}^2+\mathcal{K})(1-\frac{\lambda}{2a^2}\mathcal{H}\xi') + (\mathcal{H}^2+\mathcal{K})(1-\frac{\lambda}{4a^2}\xi'') = 0 \tag{8}$$

$$\varphi'' + 2\mathcal{H}\varphi' + \frac{3\lambda}{2a^2}(\mathcal{H}^2+\mathcal{K})\mathcal{H}'\xi_{,\varphi} = 0 \tag{9}$$

Here,the conformal Hubble parameter

$$\mathcal{H} := \frac{a'}{a} \tag{10}$$

is used and prime ($'$) denotes differentiation with respect to conformal time η. We denote the derivative with respect to the physical time t as dot ($\dot{}$), and usual Hubble parameter as $H := \frac{\dot{a}}{a}$. Since the action (4) is invariant under the change of φ's sign, we can choose the initial condition of φ so that φ increases in time (at least near the initial point). Using

$$\xi = \frac{1}{2}\varphi^2 \tag{11}$$

instead of (3), which is a good approximation near $\varphi = 0$, the solutions of the equations of motion which continue to Friedmann like phase ($H > 0$, $\dot{H} < 0$) at $t = 0$ are shown in the fig.1. In these solutions φ increases monotonically, so larger φ corresponds to later time. As can be seen from the figure, there are two classes of such solutions — singular and non-singular solutions. Solutions a and b in the fig.1 are singular at $\varphi = 0$, whereas c, d, e and f continues beyond the initial singularity. One may also notice that these non-singular solutions approaches de Sitter like solution ($H \simeq const \neq 0$) as $\varphi \to -\infty$, $t \to -\infty$. This is peculiar to the form of the potential (11). If we use (3), the behavior of the solutions near $\varphi = 0$ are quite similar to those shown in the fig.1, but H finally approaches zero in the infinite past. Although the asymptotic behavior is quite different, we will use (11) as ξ in the actual numerical calculation since it is simple, and we are only interested in the behavior of the solution near the singularity.

3 Perturbative analysis

3.1 Method of perturbation

We will consider perturbation of our model(4) to analyze the stability of the non-singular nature. To avoid the gauge ambiguity, we here use the gauge-invariant perturbation method [14]. The metric is decomposed into the back ground part and the perturbed part, and the perturbed part into scalar-, vector-, and tensor-perturbation part:

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta^S g_{\mu\nu} + \delta^V g_{\mu\nu} + \delta^T g_{\mu\nu}. \tag{12}$$

Also, the scalar field φ (or the potential ξ) is decomposed into its background and perturbed part:

$$\begin{aligned} \varphi &= {}^{(0)}\varphi + \delta\varphi, \\ \xi &= {}^{(0)}\xi + \delta\xi. \end{aligned} \tag{13}$$

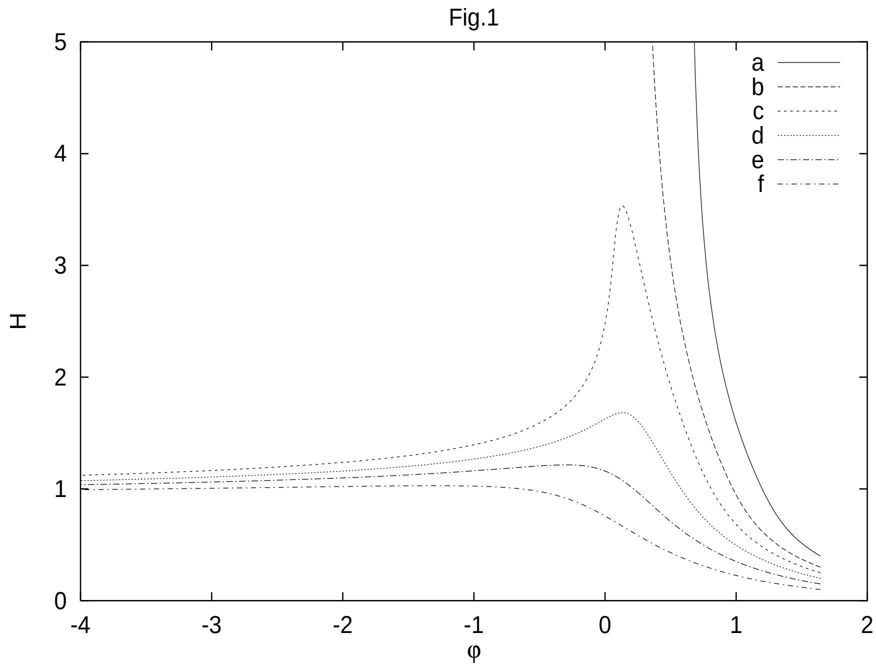

Figure 1: Behavior of Hubble parameter versus scalar field φ. Time flows from left to right since in these solutions $\dot{\varphi}$ is always positive. There are two classes of solutions: singular solutions (a and b), and non-singular solutions (c,d,e,f). We can see that the initial singularity is avoided when φ crosses zero. For detailed discussions see [13].

Change of perturbed values under gauge transformations is given by Lie derivatives as:

$$\delta Q \rightarrow \delta Q + \pounds_X {}^{(0)}Q. \tag{14}$$

We will introduce such combinations of perturbed variables that the effects of gauge transformation cancel. These gauge invariant variables are used to describe the dynamics of the perturbation. Equations of motion for perturbed variables are obtained as perturbed Einstein-like equation:

$$\delta G^\mu{}_\nu = \delta T^\mu{}_\nu + \frac{\lambda}{2}\delta K^\mu{}_\nu. \tag{15}$$

Each term is also decomposed into scalar, vector, and tensor part. Perturbed Klein-Gordon equation is not necessary since it is derived from (15).

3.2 Scalar part

Scalar part of the perturbed metric is written in terms of 4 scalar functions ϕ, ψ, B, and E,

$$\delta^S g_{\mu\nu} = a^2(\eta)\begin{pmatrix} -2\phi & B_{|i} \\ B_{|j} & -2(\psi\gamma_{ij} - E_{|ij}) \end{pmatrix}, \tag{16}$$

and following combination of gauge-dependent variables are shown to be unchanged under gauge transformations:

$$\begin{aligned} \Phi &= \phi + \frac{1}{a}[a(B-E')]', & (17)\\ \Psi &= \psi - \frac{a'}{a}(B-E'), & (18)\\ \delta\varphi^{(gi)} &= \delta\varphi + {}^{(0)}\varphi'(B-E'), & (19)\\ \delta\xi^{(gi)} &= \delta\xi + {}^{(0)}\xi'(B-E'). & (20) \end{aligned}$$

Introducing a variable

$$\Theta := \Psi + \frac{\lambda}{2a^2}\{\frac{1}{2}(\mathcal{H}^2+\mathcal{K})\delta\xi^{(gi)} - \mathcal{H}\xi'\Psi\}, \tag{21}$$

scalar part of the perturbed Einstein equation (15) gives two independent equations:

$$\begin{aligned}
&\nabla^2\Theta + \left[\frac{3\lambda\xi'}{4a^2\alpha}(\mathcal{H}^2+\mathcal{K}) - 3\mathcal{H}\right]\Theta' \\
&+ \left[-6\mathcal{H}' - 12\mathcal{H}^2 - 6\mathcal{K} - \frac{3\lambda\xi'}{2a^2\alpha}\mathcal{H}(2\mathcal{H}'+\mathcal{H}^2+\mathcal{K}) + \frac{3\lambda^2\xi'^2}{8a^4\alpha^2}(\mathcal{H}^2+\mathcal{K})\mathcal{H}'\right]\Theta \\
&- \frac{3\lambda^2\xi'}{16a^4\alpha}(\mathcal{H}^2+\mathcal{K})^2\delta\xi^{(gi)\prime} - \frac{1}{2}\varphi'\delta\varphi^{(gi)\prime} \\
&+ (\mathcal{H}^2+\mathcal{K})\left[\frac{3\lambda}{4a^2}\mathcal{H}' + \frac{3\lambda}{4a^2\alpha}(3\mathcal{H}'+2\mathcal{H}^2+2\mathcal{K}) - \frac{3\lambda^3\xi'^2}{32a^6\alpha^2}(\mathcal{H}^2+\mathcal{K})\mathcal{H}'\right]\delta\xi^{(gi)} = 0
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\Theta' + \left[\mathcal{H} + \frac{\lambda\xi'}{4a^2\alpha}(4\mathcal{H}'+3\mathcal{H}^2+3\mathcal{K})\right]\Theta \\
&-\frac{3\lambda^2\xi'}{16a^4\alpha}(\mathcal{H}^2+\mathcal{K})(2\mathcal{H}'+\mathcal{H}^2+\mathcal{K})\delta\xi^{(gi)} - \frac{1}{2}\varphi'\delta\varphi^{(gi)} = 0
\end{aligned} \tag{23}$$

where α is

$$\alpha = 1 - \frac{1}{2}\lambda\mathcal{H}\xi'. \tag{24}$$

Eliminating $\delta\varphi^{(gi)}$ from (22) and (23), we can obtain one wave function for Θ. However, for finding numerical solutions it is more convenient to directly integrate (22) and (23). Decomposing Θ into Fourier mode and using (11) for ξ, we obtain the numerical solutions shown in fig.2. Although the amplitude of the perturbation is somewhat enhanced near the peak of Hubble parameter, there is no growing mode appearing in the metric perturbation.

Fig. 3 shows the effective density contrast defined by:

$$\frac{\delta\rho}{\rho} := \frac{\delta^S G^0{}_0}{G^0{}_0}. \tag{25}$$

We notice that the small scale effective density contrast grows in Friedmann phase. In late time in Friedmann phase, the effect of the Gauss-Bonnet term becomes negligible and the evolution of the density contrast can be considered as in the ordinary Friedmann universe. Thus the growth of the density contrast is the consequence of the free scalar field remaining in the Friedmann universe. This result is analytically understood as follows. Putting $\lambda = 0$ in equations (22) and (23), we have a equation for Θ:

$$\Theta'' - \nabla^2\Theta + 6\mathcal{H}\Theta' + (10\mathcal{H}' + 20\mathcal{H}^2)\Theta = 0. \tag{26}$$

Using back ground equations (7)(8)(9) with $\lambda = 0$, and introducing

$$u = \frac{a\Theta}{\varphi'} \tag{27}$$

$$\theta = \frac{\mathcal{H}}{a\varphi'}, \tag{28}$$

the equation can be written in a simple form[14]:

$$u'' - \nabla^2 u - \frac{\theta''}{\theta}u = 0 \tag{29}$$

Regarding the perturbation as plane wave, the solution for this equation can be easily found in both large and small wave number limits. For large enough wave number ($k^2 \gg \theta''/\theta$), the solution of the equation(29) is

$$u \propto \exp(\pm ik\eta).$$

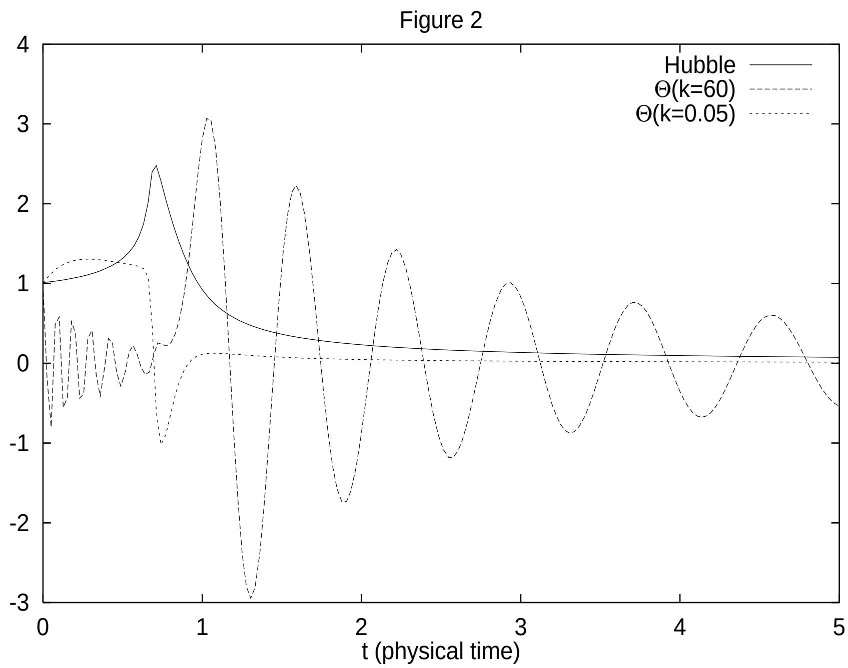

Figure 2: Behavior of scalar perturbation. Perturbations with different wave numbers are shown here, and there is no growing mode.

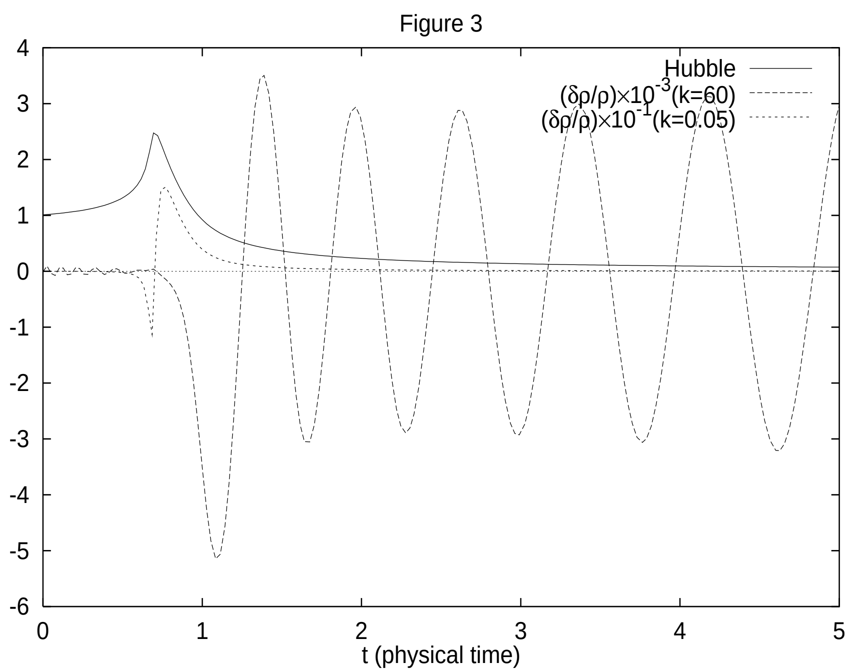

Figure 3: Density contrast calculated using the solution shown in Fig.2. Small scale density contrasts grow as $\frac{\delta\rho}{\rho} \propto t^{1/3}$ due to the free scalar field in the Friedmann universe.

Transforming this back into the gauge-invariant perturbative variables, for density contrast we have

$$\frac{\delta\rho}{\rho} \propto a \propto \eta^{1/2} \propto t^{1/3}. \tag{30}$$

Thus the density contrast grows like $t^{1/3}$.

3.3 Vector part

The perturbed metric for vector perturbation is defined as:

$$\delta^V g_{\mu\nu} = a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_j & F_{i|j} + F_{j|i} \end{pmatrix} \tag{31}$$

Vector function S_i and F_i satisfy $S_i{}^{|i} = F_i{}^{|i} = 0$, and the combination $P_i = S_i + F_i'$ is unchanged under the gauge transformation. Vector part of the perturbed Einstein-like equation (15) gives 2 non-trivial equations:

$$(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi')(\frac{1}{2}\nabla^2 + \mathcal{K})P_i = 0, \tag{32}$$

$$\{(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi')(P_{i|j} + P_{j|i})\}' + 2\mathcal{H}(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi')(P_{i|j} + P_{j|i}) = 0. \tag{33}$$

Equation (32) shows the spatial behavior of the vector perturbation, indicating the fluctuation of curvature scale. Equation (33) can be easily integrated to give

$$P_{i|j} + P_{j|i} \propto a^{-2}(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi')^{-1} = \frac{1}{a^2\alpha}. \tag{34}$$

which indicates that the vector perturbation decreases as the universe expands.

3.4 Tensor part

The tensor part of the metric perturbation is defined as:

$$\delta^T g_{\mu\nu} = a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}, \tag{35}$$

which satisfies the constraint $h^i{}_i = 0$, $h_{ij}{}^{|j} = 0$. Tensor part of (15) gives one non-trivial equation for i-j part:

$$\{(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi')h^i{}_j{}'\}' + 2\mathcal{H}(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi')h^i{}_j{}' - (1 + \frac{\lambda}{2a^2}\mathcal{H}\xi' - \frac{\lambda}{2a^2}\xi'')(\nabla^2 - 2\mathcal{K})h^i{}_j = 0$$

Expanding h_{ij} with transverse-traceless basis tensors as

$$h_{ij} = h_+(k,\eta)\mathbf{e}_{+ij}(k) + h_\times(k,\eta)\mathbf{e}_{\times ij}(k). \tag{36}$$

for each polarization mode the function $h(k)$ satisfies the equation of motion using the physical time:

$$\ddot{h} + (3H + \frac{\dot{\alpha}}{\alpha})\dot{h} + \frac{2\mathcal{K} + k^2}{a^2\alpha}(1 - \frac{\lambda}{2}\ddot{\xi})h = 0 \tag{37}$$

Numerical solutions for $h(k)$ with different wave numbers ,in the same background as scalar and vector perturbations, are shown in the fig.4. There is an instability in the de Sitter-like phase, which results from large positive value of $\ddot{\xi}$ in the equation (37).

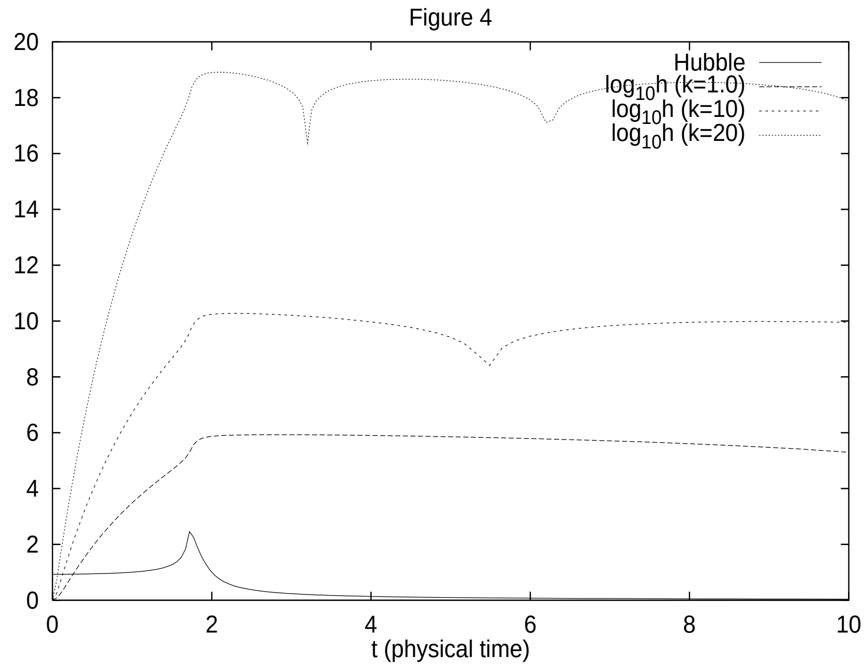

Figure 4: Tensor perturbation in the same background. Note that the tensor perturbations are plotted with logarithmic scale. Growing mode appears in the initial de Sitter-like phase. The smaller the perturbation scale is, the larger the growth rate becomes.

4 Conclusion

We have made a perturbative analysis of the cosmological model presented by Rizos and Tamvakis [13], to investigate the stability of the model. The perturbative equations of motion are solved numerically, and we found that the system is unstable under tensor perturbation. This instability appears in the de Sitter-like stage, and kinetic-driven inflation is thought to be responsible for the growing mode of the perturbation. Since the form of the function ξ coming from superstring effective action gives different asymptotic behavior of H, from this calculation alone we cannot conclude that this instability continues from the infinite past. But anyway, even with the modulus-geometry coupling of the form (3), there exists a tensor-part instability at least right before the Hubble parameter peak. Taking this model more seriously, we have to point out some problems. First of all, we have to check the validity of the effective action (1). Since the inclusion of the 1-loop effect predicts somewhat different character of the solution, it is certainly possible that the higher loop correction might change the nature of the cosmological model drastically. In this analysis we have ignored the effects of dilaton, but of course we must take these into acount. Viewing this model as a realistic model of our universe, we also have to find a mechanism to create the ordinary matter, i.e. there have to be something like reheating in the inflationary universe.

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